Treasury Auctions: The Spanish format[*]

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Abstract. The Spanish Treasury is the only Treasury in the world that uses a hybrid system of discriminatory and uniform price auctions to sell government debt: winning bidders pay their bid price for each unit if this is lower than the weighted average price of winning bids, and pay the weighted average price of winning bids otherwise. Following Gordy (1996), we model the Spanish auction as a common value auction of multiple units with private information, allowing for multiple bids. Simulations show that bidders use bid spread to hedge against both uncertainty and the winner's curse, and that the expected seller's revenue is higher in the Spanish than in the discriminatory auction in a number of cases.

Resumen. El Tesoro español es el único Tesoro del mundo que utiliza un sistema híbrido de subasta discriminatoria y uniforme para vender deuda del estado: los pujadores ganadores pagan su puja por cada unidad, si esta fue menor que la media de las pujas ganadoras, y pagan la media de las pujas ganadoras en caso contrario. Siguiendo a Gordy (1996), desarrollamos un modelo para la subasta española, como una subasta de valor común y múltiples unidades, con información privada, permitiendo que los pujadores pujen por varias unidades. Utilizando simulaciones, encontramos que un pujador puja por distintas unidades a precios distintos, para protegerse tanto de la incertidumbre sobre el valor del bien como de la maldición del ganador, y que el ingreso esperado del vendedor es mayor con la subasta española que con la discriminatoria en varios casos.

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1 Introduction

One of the most important auction markets in the world is the market for government debt. Treasuries apply mainly two auction formats: discriminatory and uniform price auctions. In a discriminatory auction, winning bidders pay their bid price for each unit. In a uniform price auction, all winning bidders pay the same price for each unit, the minimum accepted price. The Spanish Treasury is the only one that uses a hybrid system of discriminatory and uniform price auctions: winning bidders pay their bid price for each unit if this is lower than the weighted average price of winning bids, whereas they pay the weighted average price of winning bids otherwise. With the Spanish format, the price that a bidder pays for certain units depends on the bids of all winning bidders, including his own bids. This fact increases the players' strategic considerations with respect to discriminatory and uniform auctions, even in the simplest models.

Most of the discussion on Treasury auction design focuses on the choice between discriminatory and uniform auctions, addressing the issue of which auction format yields the seller higher revenue. Neither theoretical nor empirical papers are conclusive. Treasuries used to favor the discriminatory auction: according to Bartolini and Cotarelli (1997), in 1994, 39 out of the 42 countries that used auctions to sell Treasury securities used discriminatory auctions and only 2 uniform auctions, Spain being the only country that used a different format. However, some Treasuries have lately changed to the uniform format: for example, the U.S. Treasury, which exclusively used the discriminatory format, experimented with the uniform auction between 1992 and 1998, and then switched entirely to the uniform auction in November 1998. Treasuries in Mexico and Italy, among others, have also changed from the discriminatory to the uniform format.

This paper considers the third auction format that is used to sell government debt, the Spanish auction. We analyze a discrete auction example with common values and with private information, for both the Spanish and the discriminatory formats. We adapt Gordy's (1996) model for the discriminatory auction to the Spanish auction. It is a discrete model that allows explicitly for the use of multiple bids in a multiple-unit auction. Using numerical simulations, we find all (if any) Bayesian Nash symmetric equilibria for the Spanish and the discriminatory auctions, for a wide range of parameter combinations, and compare equilibrium bids and seller's revenue for both auction formats.

Many authors that study multiple-unit auctions where bidders demand more than one unit, follow the "share auctions" approach, proposed by Wilson (1979), where the good is assumed to be perfectly divisible and a bid is a smooth demand schedule. In this paper we depart from this continuous approximation: the model that we use is discrete. There are two units for sale, and bidders submit one price bid for each of the two units; prices are restricted to a finite set. We use a discrete model mainly for two reasons. Firstly, because in the case of Treasury auctions a discrete model provides a better description of reality than a continuous model, given that there are minimal increments of prices and quantities in actual auctions. And secondly, because as Gordy (1996) points out, the advantage of a discrete framework is that "it makes a difficult problem solvable".

Note that the choice of a discrete or a continuous model is not innocuous. Firstly, because discrete and continuous models have qualitatively different equilibria. The intuition is as follows. If bidders submit a continuous downward sloping demand, a collusive equilibrium with a low stop-out price can be sustained: at such an equilibrium, in order to capture a unit increase in quantity, a bidder inevitably bids up the price, so that his total profit will fall. However, if bidders can only submit a finite number of bids, this is no longer true: a bidder can capture a "large" increase in quantity with a negligible increase in price, with positive probability, and therefore competition increases.[1] Secondly, since the choice of a discrete or a continuous model affects the existence of equilibria. For example, Haller and Lengwiler (1998) prove the existence of equilibrium in a multi-unit discriminatory auction in a discrete model and conclude that non-existence is an artifact of the Menezes and Monteiro (1995) model, which considers prices and quantities as continuous variables. McAdams (2001) confirms this idea: he proves the existence of pure strategy equilibrium in the uniform-price auction when bidders have multi-unit demand and values that are neither private nor common, and points out that discreteness is essential to his results: without discreteness, an equilibrium may not exist.

Our main findings can be summarized as follows. Firstly, bidders spread their bids more in the Spanish than in the discriminatory auction. There are two contributing factors. On the one hand, bidders bid more aggressively on the first units in the Spanish than in the discriminatory auction, since they have a lower expected cost of doing so: if they win with the highest bids, they only pay the weighted average price of winning bids instead of their bids, as they do in the discriminatory auction. On the other hand, in the Spanish auction bidders have incentives to lower their bids on later units, since they might determine the price they pay for earlier units. Secondly, it is not possible to offer a complete ranking of the Spanish and the discriminatory auctions in terms of expected seller's revenue, both because the ranking varies with the value of the parameters, and because for some parameter values there are multiple equilibria, and the ranking

[1] See Nyborg (2001).

depends upon which of the equilibria is examined. Nevertheless, the Spanish auction provides higher expected revenue than the discriminatory auction in a number of cases.

This paper is organized as follows. Section 2 presents a survey of related literature; Section 3 presents the model; Section 4 presents some features of the algorithm used to solve the model and a discussion of the parameters used; Section 5 presents the results; and finally, Section 6 concludes the paper.

2 Survey of the literature

An abundant literature exits for the auctioning of a single, indivisible item, in uniform and discriminatory auctions, and general results are established, results that can be extended to settings with multiple units, if each bidder has a taste for only one item.[2] But as Ausubel and Cramton (2002) mention, "in environments where bidders desire multiple items, general results (...) are not well understood". The reason is that the problem is very complex. Firstly, because bidders have a very large strategy space. And secondly, because there is a strategic component in bidding: for example, in a uniform auction, bids on later units might determine the price the bidder pays for earlier units. In addition, if there is private information, as is usually assumed in Treasury auctions, equilibrium bids must address not only the strategic component of bidding, but also the inference problem due to asymmetric information, and the problem becomes even more complex.

According to Viswanathan et al. (2001), "while there has been progress in uniform price auctions and in optimal auctions, very little is known about the multi-unit discriminatory auction with variable awards". For discrete units and private values, Engelbecht-Wiggans and Kahn (1998) solve the two unit discriminating auction numerically; Tenorio (1999) solves a two bidders and two units example; and Reny (1999) shows the existence of equilibrium with downward sloping marginal valuations. For discrete units and common value, with private information, Gordy (1996) uses numerical simulations to find equilibria for the discriminatory auction, when two units of an indivisible good are auctioned to I bidders. He finds evidence that supports the conjecture that multiple bidding can be used to hedge against uncertainty and the winner's curse, when bidders are risk averse. For continuous units, Viswanathan et al. (2001) analyze the two bidder discriminating auction with affiliated valuations; Wang and Zender (2002) characterized the equilibria for both the discriminatory and the uniform auction when bidders possess private information; and Ausubel and Cramton (2002) provide several examples to demonstrate that auctions results are inefficient, and that the ranking of uniform and discriminatory auctions in terms of

[2] See for example Milgrom and Weber (1982) for a characterization of auctions of one indivisible unit, and Harris and Raviv (1981) and Weber (1982) for a characterization of auctions on multiple units, where bidders only demand one unit.

expected seller's revenue is ambiguous: they provide examples with reasonable specifications of demand where the uniform auction provides higher expected seller's revenue than the discriminatory auction, and equally reasonable specifications where the reverse is true.

If little is known about the multi-unit discriminatory auction, even less is known about the Spanish auction. To our knowledge, the properties of the Spanish auction have been studied only by Salinas (1990), Mazón and Nuñez (1999), Álvarez et al. (2002) and Abbink et al. (2001). Salinas (1990) presents a model where demand is restricted to one unit per bidder, and values are private. He uses the results of Maskin and Riley (1989) to argue that the Spanish mechanism generates the same expected revenue as uniform and discriminatory auctions. Mazón and Nuñez (1999), in a model that assumes that demand functions are common knowledge and a bid is a price-quantity pair, show that the Spanish auction is the equivalent in terms of revenue to the seller to the discriminatory format, and that both formats maximize the seller's revenue. They also present an empirical analysis, using data of Spanish security auctions between 1993 and 1997. They find evidence of the good functioning of the market, and the relatively low price differentials paid by accepted bids, which is consistent with the predictions of the model. Álvarez et al. (2002) follow the "share auctions" approach proposed by Wilson (1979), where the good is assumed to be perfectly divisible and a bid is a smooth demand schedule, and solve a common value model with no private information. They use the model in Wang and Zender (2002) to characterize the set of linear equilibria for the Spanish auction. For that setting, they find parameter values in which the Spanish format yields higher expected seller's revenue than the other two auction formats. Finally, Abbink et al.(2001) report an experiment that compares the discriminatory and the Spanish auction, in a common value model with private information. Results show significantly higher seller's revenue with the Spanish format, and the use of bid spreading in both auction formats.

3 The model

To analyze the Spanish auction, we adapt Gordy's (1996) model, which studies the discriminatory auction, to the Spanish format. Two indivisible and identical units are for sale, and $I \geq 2$ bidders compete for them. Each bidder submits two sealed bids, specifying a price, but not a particular unit.[3] The two units are awarded to the two highest bids, and if there is a tie, there is randomization among the tied bids.

[3] Bartolini and Cottarelli (1997) report that of the 34 countries in their study, only two (Latvia and Jordan) restrict bidders to a single bid each. The great majority place no limit on the number of bids per bidder. Besides, bidders do make use of multiple bids: Gordy (1996) reports that bidders in the United States and Portugal make an average of three bids per bidder, and Mazón and Nuñez (1999) report that bidders in Spain make an average of 2.6 bids per bidder.

The true value of each unit of the good for sale, v, is unknown to the bidders at the time of bidding. Let $F(v)$ be the prior distribution of v. We assume that $F(v)$ is public information and is $Beta(\alpha\mu; \alpha(1-\mu))$; this distribution has a mean μ and variance decreasing in α.[4] Bidders have private information: each bidder observes a signal from the finite set $X := \{0; 1; \ldots; K\}$, with $K > 0$. Signals are independent across bidders and each bidder only observes his own signal. The probability distribution of the signal conditional on v is assumed to be $Binomial(K; v)$.

Bidders combine public information, the prior on v, and private information, the signal received, using Bayes' rule. The beta distribution is conjugate for a signal drawn from a binomial distribution[5], so that the posterior distribution of v, $F(v/x)$, is $Beta(x + \alpha\mu; K - x + \alpha(1-\mu))$, where x is the signal that the bidder has received ($x \in X$). The posterior expectation of v is $E(v/x) = \delta(x/K) + (1-\delta)\mu$, where $\delta = (1 + \alpha/K)^{-1}$. Thus, $E(v/x)$ is a strictly convex combination of the estimation of v based on public and private information, μ and x/K respectively, where the weight given to each one depends on its relative accuracy. Note that the posterior variance of v is decreasing in K and α.

We allow for a finite number of prices: prices are restricted to a finite set $\Lambda := \{0; 1/\lambda; 2/\lambda; \ldots; 1\}$, where λ is some positive integer. As argued in the Introduction, the choice of a discrete model is not innocuous and in practice, Treasury auctions in Spain (and in most countries) have restrictions on the set of bids permitted.[6]

We assume that bidders are risk averse[7], and that they have a constant absolute risk aversion (CARA) utility function, $U(z) = -\exp(\rho z)$, where $\rho > 0$ is the coefficient of absolute risk aversion, common to all bidders, and z is the profit obtained by the bidder on the auction, which depends on the auction format.

We consider two auction formats: the Spanish auction and the discriminatory auction. Profits to bidders from the auction are calculated as the valuation minus payments. For both auction formats, the valuation is equal to $2v$ if a bidder gets two units, since he values both units equally; and is equal to v if he only gets one unit. Payments depend on the auction format. In the Spanish auction, winning bids pay the bid price for each unit if this is lower than the weighted average price of winning bids (WAP), and pay the WAP if the bid

[4] The beta distribution can take on a variety of unimodal and bimodal forms with support in $[0; 1]$, and is well suited for modeling the distribution of the true value of the good.

[5] See DeGroot (1970).

[6] Bids must be made for at least 1,000 euros, and bids for larger amounts must be made for a multiple of 1,000 euros. Investors must indicate the desired nominal amount and the price they are willing to pay. The price must be expressed as a percentage of the nominal (or face) value. For additional imformation on the Spanish auction rules see http://www.mineco.es/tesoro/

[7] See Gordy (1996), (1999) and Wang and Zender (2002) for a justification.

price is equal to or higher than the WAP. In the discriminatory auction, winning bids pay the bid price. For example, assume that there are two bidders, A and B, bidding $(4;1)$ and $(3;2)$, respectively. Winning bids are 4 and 3, and each bidder gets one unit in both auction formats. In the Spanish auction, A pays $(4+3)/2$, the WAP, and B pays 3. In the discriminatory auction, A pays 4 and B pays 3.

The model is a simultaneous game of incomplete information and we consider pure strategy Bayesian Nash equilibrium. A strategy or a decision rule for bidder i is a function s_i, which gives the player's pair of bids for each realization of the signal he received, x_i. Therefore, $s_i : X \to \Lambda \times \Lambda$, specifies for each signal $x_i \in X$ a pair of bids $s_i(x_i) = \left(s_i^1(x_i); s_i^2(x_i)\right) \in \Lambda \times \Lambda$, where $s_i^1(x_i) \geq s_i^2(x_i)$; we refer to $s_i^1(x_i)$ and $s_i^2(x_i)$ as the high and the low bid for bidder i and signal x_i, respectively. Let $s_{-i}(x_{-i}) \in (\Lambda \times \Lambda)^{I-1}$ be a pair of bids for all players but i, where $x_{-i} \in X^{I-1}$. Since bidder i's profit, given the auction format, only depends on $(s_i(x_i); s_{-i}(x_{-i}))$, we write his utility function as $U_i(s_i(x_i); s_{-i}(x_{-i}))$. A profile of decision rules $(s_1; \ldots; s_I)$ is a Bayesian Nash equilibrium if and only if, for all i and $x_i \in X$,

$$E\{U_i(s_i(x_i); s_{-i}(x_{-i})) \mid x_i\} \geq E\{U_i(\hat{b}_i; s_{-i}(x_{-i})) \mid x_i\}$$

for all $\hat{b}_i \in \Lambda \times \Lambda$, where the expectation is taken over v, the value of the good, and x_{-i}, the vector of his rivals' signals, and is conditional on x_i. That is, the bid pair for any signal maximizes the bidder's conditional (on the signal) expected utility, given the strategies of all other bidders. In addition, a profile of decision rules is symmetric if all bidders play the same decision rule. We only consider symmetric equilibria.[8] In such equilibria, if two bidders submit a different pair of bids, it is because they have received different signals.

4 Solving the model

We solve the model numerically, given the difficulties in solving the model analytically, even for the simplest case (two bidders, two signals and two possible prices). We find equilibria for different vectors of parameter values. The code has been written in TurboPascal 5.5.[9] In this section, we present some basic features of the algorithm we use and discuss the parameters used to calculate the equilibria.

For a given vector of parameter values, denote by Γ the set of all possible functions $s : X \to \Lambda \times \Lambda$. As both X and Λ are finite, so is Γ. In order to shorten computation time, we consider only strategies in which $s^1(x_i)$ is non-decreasing

[8] In what follows, we will drop the subindex refering to the bidder in any equilibrium strategy.

[9] The code will be provided by the authors upon request.

in x_i.[10] Denote by $\hat{\Gamma}$ the subset of Γ containing such strategies. For every $s \in \hat{\Gamma}$, we check whether $(s, \ldots, s)$ is a symmetric Bayesian Nash equilibrium: we list all the elements in $\hat{\Gamma}$ in some arbitrary order, and run through the list checking every element.

How do we check whether $s \in \hat{\Gamma}$ is an equilibrium? We extend Gordy's (1996) analysis to the Spanish auction (he considers the discriminatory auction). For a given $s \in \hat{\Gamma}$, we assume that all bidders but i are playing s, that is, $s_{-i} = (s, \ldots, s)$, and check whether bidder's i best response to s_{-i} is s. If the answer is yes, the profile of decision rules $(s, \ldots, s)$ is an equilibrium. We check all strategies in $\hat{\Gamma}$ exhaustively.

To calculate the expected utility for bidder i of bidding an arbitrary strategy $\hat{s}$, given that all other players are bidding $s_{-i} = (s, \ldots, s)$, conditional on signal x_i, we proceed in the following way. Let Ω be the set of events that might occur to a bidder under the Spanish format, and let σ be an element in Ω. The elements in Ω are: (a) to get two units, and pay the WAP for the first unit and the low bid for the second unit; (b) to get one unit and pay the high bid, (c) to get one unit and pay the WAP; and (d) to get zero units. We can write:

$$E\{U_i(\hat{s}, s_{-i}(x_{-i})) \mid x_i\} = \sum_{\sigma \in \Omega} E\{U_i(\hat{s}, s_{-i}(x_{-i})) \mid x_i, \sigma\} \Pr(\sigma \mid \hat{s}, s_{-i}, x_i)$$

where $\Pr(\sigma \mid \hat{s}, s_{-i}, x_i)$ is the probability of event σ conditional on $(\hat{s}, s_{-i}, x_i)$.

Proposition 1 presents some simplifications to compute each of the terms in the previous summation. To simplify notation, let:

$V_i(\hat{s}, s_{-i}, x_i, \sigma) := E\{U_i(\hat{s}, s_{-i}(x_{-i})) \mid x_i, \sigma\} \Pr(\sigma \mid \hat{s}, s_{-i}, x_i)$

and $\Omega := \{a, b, c, d\}$, where the elements in Ω are as listed above.

Proposition 1. For the Spanish format, the following holds:

(i) $V_i(\hat{s}, s_{-i}, x_i, a) = -\sum_{x_{-i}} p^a(\hat{s}, s_{-i}, x_{-i}, x_i)\, z^a(x_{-i}, x_i)$

(ii) $V_i(\hat{s}, s_{-i}, x_i, b) = -\exp\left(\rho\left(1 - \hat{s}^1\right)\right) \sum_{x_{-i}} p^b(\hat{s}, s_{-i}, x_{-i}, x_i)\, z^b(x_{-i}, x_i)$

(iii) $V_i(\hat{s}, s_{-i}, x_i, c) = -\frac{1}{\bar{s}} \exp\left(\rho\left(1 - \frac{1}{2}\hat{s}^1 + \bar{s}\right)\right) \sum_{x_{-i}} p^c(\hat{s}, s_{-i}, x_{-i}, x_i)\, z^c(x_{-i}, x_i)$

(iv) $V_i(\hat{s}, s_{-i}, x_i, d) = -\exp\left(\rho\left(2 - \frac{3}{2}\hat{s}^1 - \frac{1}{2}\hat{s}^2\right)\right) \sum_{x_{-i}} p^d(\hat{s}, s_{-i}, x_{-i}, x_i)\, z^d(x_{-i}, x_i)$

where $p^\sigma(\hat{s}, s_{-i}, x_{-i}, x_i)$ denotes the probability that event σ takes place conditional on $(\hat{s}, s_{-i}, x_{-i}, x_i)$ and $z^\sigma(x_{-i}, x_i)$ is defined in the Appendix.

This proposition is straightforward from Gordy (1996), and thus we omit the proof. The interest of the proposition is to decouple, for instance, $V_i(\hat{s}, s_{-i}, x_i, b)$ in (ii), into two terms. The first, $-\exp\left(\rho\left(1 - \hat{s}^1\right)\right)$, depends only on $\hat{s}$. The

[10] Gordy (1996) looks for equilibria within Γ and in every equilibria he obtains for the discriminatory auction, the high bid is non-decreasing in the signal, as we are assuming.

second term is a summation across the possible vectors of rivals' signals, x_{-i}, which is a summation of a finite number of terms. Furthermore, each term in this summation is a product: $p^b(\hat{b}; s_{-i}; x_{-i}; x_i)\, z^b(x_{-i}; x_i)$, where $p^b(\hat{b}; s_{-i}; x_{-i}; x_i)$ is easily computable and $z^b(x_{-i}; x_i)$ does not depend on strategies, and thus only has to be computed once for every vector of parameter values. Note that Proposition 1 states implicitly that the expectation over v is embedded in the latter product.

The algorithm that we use differs from Gordy's (1996). He starts with an arbitrary $s \in \Gamma$, assumed to be played by $I - 1$ bidders, and computes the best response to it (in Γ) by the remaining bidder, say $f(s)$. If $f(s) = s$, then s is an equilibrium and another element in Γ is selected as a candidate at random. If $f(s) \neq s$, then the best response to $f(s)$ is computed, $f(f(s))$, and the process is repeated until some strategy s^0 is the best response to itself ($f(s^0) = s^0$). Note that Gordy's exploration of Γ might enter a loop if $f(f(s)) = s$ but $f(s) \neq s$.

The vector of parameters of the model is $(\mu; \alpha; \rho; I; \lambda; K)$, and we follow Gordy (1996) mostly in the selection of the values of the parameters we use.[11] We have kept μ, the *a priori* expected value of the value of the good for sale, equal to 0.75 throughout all the vectors of parameter values considered. Note that μ must lie in $(0; 1)$, and that if μ were too small, the *a posteriori* expected value of v would be too small for bidders to bid strictly positive bids for every possible signal. For the parameter of accuracy of prior information, α, we have selected values in $\{2; 4; 6; \ldots; 20\}$. For $\alpha = 2$, i.e., the maximum uncertainty case within the previous set, the standard deviation of the prior distribution of v is 0.25. For $\alpha = 20$, the standard deviation is approximately 0.1. For the risk aversion parameter, ρ, we have selected values in $\{1; 5; 10\}$. The remaining parameters, I, the number of bidders, λ, where $\lambda + 1$ is the number of prices the bidder can choose from, and K, where $K + 1$ is the number of possible signals, affect the computation time. To keep this computation time within an acceptable limit, we take I in $\{2; 3; 4\}$ and $(\lambda; K)$ in $\{(5; 2); (5; 4); (9; 2)\}$. In summary, for each auction format we have to explore 270 parameter combinations (10 values for $\alpha \times 3$ for $\rho \times 3$ for $I \times 3$ combinations of λ and K).

Figure 1[12] shows a strategy that is an equilibrium for $(\alpha; \rho; I; \lambda; K) = (4; 1; 2; 5; 4)$. The vector of possible signals is represented on the horizontal axis, and the vector of possible prices on the vertical axis. Since K is equal to 4, there are 5 possible signals, $\{0; 1; 2; 3; 4\}$, and since λ equals 5, permitted bids are $\left\{0; \frac{1}{5}; \frac{2}{5}; \frac{3}{5}; \frac{4}{5}; 1\right\}$. For each signal, we plot the high (circle) and low (square) bid.

[11] Gordy (1996) searches for equilibria in more than 1,000 parameter combinations, and presents results for a subset of them, mentioning that results for other parameter values are broadly consistent. We are constrained by computation time, given that we exhaustively check whether all strategies in $\hat{\Gamma}$ are an equilibrium, and are therefore able to consider a smaller number of parameter combinations than Gordy (1996).

[12] At the end of the paper.

5 Results

In this section we present the results about the bid spread and seller's revenue for both auction formats.

When considering the equilibria, an initial issue that arises is that of existence. There exists at least one equilibrium for 51% of the parameter combinations considered for the Spanish auction, and for 96% for the discriminatory auction. Furthermore, the equilibrium is unique in 60% of the parameter combinations for which there is equilibrium for the Spanish auction, and in 41% for the discriminatory auction. Therefore, our paper adds evidence to the non-existence and non-uniqueness of equilibria. In the rest of the paper, we concentrate on parameter combinations for which there exists at least one equilibrium for both the Spanish and the discriminatory formats. There are 136 such combinations (out of the 270 initially considered).

5.1 Bid Spread

In this subsection we investigate how bidders use the fact that they can bid for each unit at a different price. Note that the bidders' value of the good is the same for both units, but that in many equilibria they bid two different prices for each unit given the same signal.

First, we compare the use of multiple bidding in the Spanish and the discriminatory auctions. For $i = 1; 2$, and strategy $s = \left(s^1; s^2\right)$, define the expected high (low) bid as $E(s^i) = \sum_{x=0}^{K} s^i(x) \Pr(x)$.[13] In the example in Figure 1, the expected high bid is 0:54, and the expected low bid is 0:46. A priori, a strategy that is an equilibrium for the Spanish auction should have a higher expected high bid and a lower expected low bid than a strategy that is an equilibrium for the discriminatory auction. A higher expected high bid, because in the Spanish auction, bidders can increase the high bid at a lower cost than in the discriminatory auction, since if they overbid, they only pay the WAP. And a lower expected low bid, since in the Spanish auction the low bid may determine the price the bidder has to pay on both units, so that bidders have an incentive to lower the low bid. Does it hold that the expected high bid is higher in the Spanish than in the discriminatory auction? And that the expected lower bid is lower in the Spanish than in the discriminatory auction?

As expected, our results show that in 92% of the 136 parameter combinations for which there are equilibria for both auction formats, the expected high bid is higher (or equal) in the Spanish than in the discriminatory auction, and in 84% of the 136 parameter combinations, the expected low bid is lower (or equal) in the Spanish than in the discriminatory case. Therefore, given the

[13] When there are multiple equilibria for a given parameter combination, we assume that all equilibria occur with the same probability and calculate the expected high (low) bid across equilibria.

rules of the auctions, bidders use a higher bid spread in the Spanish than in the discriminatory auction, because they bid higher for the first unit and lower for the second unit than in the discriminatory auction.

Second, we test whether Gordy's (1999) conjectures for the discriminatory auction about the use of bid spread also hold for the Spanish auction. Gordy (1999) makes two conjectures about why bidders submit multiple bids in a pure common-value auction. His first conjecture is that risk averse investors will have a downward sloping demand for any risky security. And his second conjecture is that multiple bids may offer a way to hedge against the winner's curse. By spreading bids over a range of prices, bidders hedge against the risk of winning due to a misestimation of the value of the security (the winner's curse), since when demand is weaker than they expected (when they have overestimated the value of the good), they win at a lower average price than expected, and when demand is stronger than expected, they win at a higher price than expected. In other words, multiple bidding reduces the winner's curse by aligning the bidder's outcome more closely to the aggregate outcome of the auction. If both conjectures are true, bidders' use of multiple bids increases with the degree of uncertainty in the market, since the greater the uncertainty, the riskier it is to buy and the greater the potential for the winner's curse. And the winner's curse conjecture implies that bidders' use of multiple bids increases with the number of bidders. Gordy (1999) finds evidence to support these conjectures in a sample of Portuguese Treasury bill auctions, and provides theoretical motivation.

A priori, this strategic use of bid spread should also be present in the Spanish auction. Therefore, as in Gordy (1996), we calculate, for any s that is an equilibrium strategy, the bid spread, $\Delta(s)$, defined as the difference between the expected high and low bid. Formally: $\Delta(s) = \sum_{x=0}^{K} \left[s^1(x) - s^2(x) \right] \Pr(x)$. Since $s^1(x)$ and $s^2(x)$ lie in $[0,1]$ with $s^1(x) \geq s^2(x)$, so does $\Delta(s)$. In the example in Figure 1, $\Delta(s) = 0.08$. Note that if Gordy's conjectures are correct, bid spread should increase as uncertainty and the potential for winner's curse increase, i.e., as the degree of risk aversion, ρ, and the number of bidders, I, increase, and as the precision of public or private information decreases, i.e., as α and K decrease.

Table 1 presents the average bid spread for different parameter combinations, for both the Spanish and the discriminatory formats. For example, the average bid spread for parameter combinations with K equal 2 for the Spanish auction is 0.148, and for the discriminatory auction 0.105.

Table 1: Average bid spread

	K		λ		ρ			I		
Parameter value	2	4	5	9	1	5	10	2	3	4
Spanish	0.148	0.104	0.114	0.190	0.039	0.116	0.201	0.074	0.173	0.221
Discriminatory	0.105	0.055	0.065	0.154	0.001	0.074	0.151	0.035	0.128	0.163
	α									
Parameter value	2	4	6	8	10	12	14	16	18	20
Spanish	0.289	0.225	0.213	0.163	0.201	0.133	0.103	0.086	0.054	0.049
Discriminatory	0.135	0.146	0.145	0.128	0.110	0.094	0.078	0.071	0.041	0.037

In Table 1, note first that for all the parameter combinations considered, as expected, the bid spread is higher for the Spanish than for the discriminatory auction: as we have argued above, the expected high bid is higher and the expected lower bid is lower in the Spanish than in the discriminatory auction.

For both auction formats, the bid spread is increasing in risk aversion, ρ, and in the number of bidders, I, and is decreasing in the accuracy of private information, K. With respect to public information, as the accuracy of public information, α, decreases, the bid spread also decreases, both for the Spanish and the discriminatory formats, except for the change from $\alpha = 8$ to $\alpha = 10$ for the Spanish, and from $\alpha = 2$ to $\alpha = 4$ for the discriminatory auction. Therefore, our results suggest that Gordy's conjectures are also valid for the Spanish auction: bidders spread their bids to hedge against uncertainty and the winner's curse.

5.2 Seller's Revenue

If the seller's objective is to maximize revenue, should he use the Spanish or the discriminatory format? In this subsection, we try to answer this question by, firstly, comparing the probability distributions of the seller's revenue in the first-order stochastic sense, secondly, comparing the expected seller's revenue for both auction formats, and thirdly, using a stronger definition of dominance.

A priori, what is the intuition about how both auction formats compare in terms of the seller's revenue? There are two opposite effects. On the one hand, as we have argued before, bidders in the Spanish auction bid more aggressively for the first unit than in the discriminatory auction, given that if they have the higher bid, they only pay the WAP; therefore, the seller's revenue could be higher for the Spanish than for the discriminatory auction. On the other hand, the fact that bidders only pay the WAP instead of their bid if they have the higher bid, and additionally, the fact that they have incentives to lower the low bid, suggest that the seller's revenue could be lower for the Spanish than for the discriminatory auction.

In order to compare the seller's revenue for both auction formats, we first built the probability distribution of the seller's revenue for each auction format and each vector of parameters, and compared them using first-order stochastic dominance.[14] The idea is that every expected utility maximizer that values more over less, given two probability distributions, prefers the one that first-order dominates the other. We built the probability distribution of the seller's revenue in the following way. Note that for both auction formats, given a vector of parameter values, the seller's revenue is uniquely determined by a strategy, s, and an $I + 1$ dimensional vector of signals, x. Denote by $r(x; s)$ the seller's revenue when bidders use strategy s and observe signals on x. Next, denote by $\Pr(r(x; s))$ the probability of $r(x; s)$. Assuming that the vector of signals and the equilibrium strategy chosen by the bidders are independent, and that whenever there are multiple equilibria, strategies are equiprobable, we have that $\Pr(r(x; s)) = \Pr(x) \Pr(s) = \Pr(x)(1/N)$, N being the number of equilibrium strategies for the given vector of parameter values. Using these probabilities, we calculate the distribution function of the seller's revenue.

When we consider first-order stochastic dominance, we can only compare probability distributions for 76 of the 136 parameter combinations for which equilibria exists for both auction formats. For the remaining 60 cases, the probability distributions are not comparable in the first-order stochastic sense. For the 76 cases for which they are comparable, the probability distribution for the Spanish auction dominates the probability distribution for the discriminatory auction in the first-order stochastic sense for 23 parameter combinations, the probability distribution for the discriminatory auction dominates the probability distribution for the Spanish auction for 37 parameter combinations, and probability distributions for both auction formats are identical for 16 parameter combinations. Therefore, neither the Spanish nor the discriminatory auction yields unambiguously higher seller's revenue: the comparison depends on the parameter combinations considered.

Since first-order dominance allows us to compare only 56% of the parameter combinations for which equilibria for both auctions exists, we next compare the expected seller's revenue. Note that if the seller is risk neutral, he will rank both auction formats considering expected revenue. Define the seller's expected revenue given strategy s, as $R(s) = \sum_x r(x; s) \Pr(x)$, where x is the $I + 1$ dimensional vector of bidders' private signals, and $r(x; s)$ is the seller's revenue given such a vector and strategy s, as defined above. Note that $R(s) \in [0; 2]$. In the example in Figure 1, $R(s) = 1.08$. Again, when there are multiple equilibria for a given parameter combination, we consider that all equilibria occur with the same probability and calculate average expected revenue accordingly.

For the 136 parameter combinations for which there are equilibria for both auctions, the Spanish auction has higher average expected seller's revenue than the discriminatory auction in 50% of the parameter combinations considered,

[14] The distribution of monetary payoffs $G_1(\cdot)$ first-order stochastically dominates the distribution $G_2(\cdot)$ if and only if $G_1(x) \geq G_2(x)$ for every x. That is, the probability of getting at least x is higher under $G_1(\cdot)$ than under $G_2(\cdot)$. See MasColell et al (1995).

the discriminatory auction has higher average expected seller's revenue than the Spanish auction in 38:2%, and the average expected seller's revenue is equal for both auction formats in the remaining 11:8%.[15] How do these results compare with our previous results on ¯rst-order stochastic dominance? Given a probability distribution $G_1(:)$ that ¯rst-order dominates distribution $G_2(:)$, the mean under $G_1(:)$ is higher than the mean under $G_2(:)$; although a ranking of means does not imply that one distribution stochastically dominates the other.[16] Therefore, our results suggest that in most parameter combinations for which probability distributions are not comparable in the ¯rst-order stochastic sense, the Spanish auction has higher expected seller's revenue than in the discriminatory auction. Once more, the comparison depends on the parameter combination considered. Note that when comparing the expected seller's revenue, the Spanish auction ranks better than the discriminatory auction in half of the cases.

How does the expected seller's revenue vary with the parameters? Table 2 presents the average expected seller's revenue for di®erent parameter combinations for each auction format. For example, the expected seller's revenue in equilibria for all parameter combinations with $K = 2$, is 1:159 for the Spanish auction and 1:119 for the discriminatory auction.

Table 2: Average expected seller's revenue

	K		¸		½			I		
Par. Value	2	4	5	9	1	5	10	2	3	4
Spanish	1.159	1.136	1.128	1.213	1.229	1.175	1.100	1.071	1.207	1.253
Discriminatory	1.119	1.121	1.090	1.187	1.158	1.157	1.069	1.034	1.177	1.218
	®									
Par. Value	2	4	6	8	10	12	14	16	18	20
Spanish	0.956	1.005	1.090	1.120	1.168	1.180	1.199	1.211	1.217	1.230
Discriminatory	0.854	0.974	1.029	1.101	1.124	1.144	1.167	1.183	1.206	1.207

From Table 2, note that for both auction formats, the average expected seller's revenue is increasing in the number of possible prices, ¸, the number of players, I, and the accuracy of public information, ®; and is decreasing in the parameter of risk aversion, ½. The results on ½, I and ® are in accordnce with the results for Milgrom and Weber's (1982) single-unit model: bidders bid more aggressively as risk aversion decreases, and as the number of players or the accuracy of public information increases, and consequently the expected seller's revenue increases. Note also that for each of the parameters considered, the expected seller's revenue is, on average, higher for the Spanish than for the discriminatory auction.

[15] The average expected seller's revenue is equal for both auction formats whenever the probability distribution of the seller's revenue is equal for both auction formats. Note that this occurs when the high bid is identical for all signals and for both auction formats, and therefore the two units will be awarded to the high bids with probability 1.

[16] See Mas-Colell et al (1995).

Next, we drop the assumption that all equilibria occur with the same probability for a given parameter combination, and define dominance as a stronger criteria to compare the expected seller's revenue for each parameter combination. For a given combination of parameters, we say that the Spanish auction dominates the discriminatory auction in terms of expected seller's revenue, if all equilibria in the Spanish auction have no lower expected seller's revenue than all equilibria in the discriminatory auction, and at least one equilibrium in the Spanish auction has higher expected seller's revenue than any in the discriminatory auction. According to this definition, the Spanish auction dominates the discriminatory auction in terms of expected seller's revenue in 39% of the parameter combinations (which, of course, does not imply that the former is dominated by the discriminatory auction in the remaining cases).

Table 3 presents results in dominance. For example, in 42% of parameter combinations with $K = 2$, the Spanish auction dominates the discriminatory auction. It is noteworthy that the Spanish auction dominates the discriminatory auction in 82% of the cases when risk aversion is low (for $\rho = 1$), and as the parameter of risk aversion increases, the percentage falls. The intuition is simple. As we explained above, bidders bid more aggressively for the first unit in the Spanish auction than in the discriminatory auction, and this effect is especially strong if bidders have low risk aversion. This more aggressive bidding gives the seller higher expected revenue in the Spanish than in the discriminatory auction. The change in dominance when the other parameters change is not clear.

Table 3: Dominance of the Spanish auction

	K		λ		ρ			I		
Par. Value	2	4	5	9	1	5	10	2	3	4
	42%	29%	41%	34%	82%	33%	24%	31%	47%	44%
	α									
Par. Value	2	4	6	8	10	12	14	16	18	20
	63%	25%	40%	27%	33%	23%	41%	29%	44%	69%

Summarizing, we have compared the seller's revenue for both the Spanish and the discriminatory auction in three different ways and arrive at the same conclusion: the comparison depends on the parameters considered. These results coincide with results reported in other papers. For example, Ausubel and Cramton (2002), who follow the "share auction" approach, provide examples that show that the ranking of the uniform and the discriminatory auction in terms of the expected seller's revenue depends on the parameters of the model. They provide examples with reasonable specifications of demand where the uniform auction dominates the discriminatory auction in terms of expected seller's revenue, and equally reasonable specifications where the opposite is true. With respect to the comparison of the discriminatory and the Spanish auction, Álvarez et al. (2002), also using a "share auction" approach, report that if the coefficient of variation of supply to competitive bidders is small, the expected seller's revenue is higher for the Spanish than for the discriminatory auction; however, if the coefficient of variation is large, the ordering of the expected seller's revenue reverses.

6 Summary and Conclusions

This paper develops a model of multiple bids in a common value auction for the Spanish auction format, following Gordy (1996), that develops the model for the discriminatory auction. The Spanish auction is a hybrid system of discriminatory and uniform price auctions: winning bidders pay their bid price for each unit if this is lower than the weighted average price of winning bids, and pay the weighted average price of winning bids otherwise. We use a discrete model, in which there are two units for sale, and bidders bid for both units. The value of the units is unknown at the time of the auction, and bidders have private information. We numerically find all (if any) the Bayesian Nash symmetric equilibria for a wide range of parameter values.

Our main results are the following. First, bid spread is higher in the Spanish than in the discriminatory auction for two reasons. On the one hand, since the cost of overbidding is lower due to the fact that a winning bidder with the highest bid only pays the weighted average price, while he pays his bid in a discriminatory auction, bidders bid more aggressively on the first unit. On the other hand, since the bid on the second unit could change the price paid on the first unit, bidders have an incentive to lower the low bid that is not present in the discriminatory auction. Second, as Gordy (1999) suggested for discriminatory auctions, bidders spread their bids in the Spanish auction to hedge against uncertainty and the winner's curse. And third, it is not possible to offer a complete ranking of the Spanish and the discriminatory auctions in terms of expected seller's revenue, since the ranking varies with the value of the parameters.

The design of Treasury auctions is important, since countries are using them to finance public debts. We think that our results suggest that Treasuries around the world should pay attention to the Spanish auction format. Opponents of the discriminatory auction, the most widely used auction format, argue that bidders shade their bids downward more in the discriminatory than in the uniform format, aware of the cost of overbidding, and that as a result, seller's revenue is lower. For example, the U.S. Treasury, when arguing in favor of the use of uniform auctions, the format that it is now using, says that "auction participants may bid more aggressively in single-price auctions. Successful bidders are able to avoid the so-called "winner's curse". (...) We estimate that more aggressive bidding has lowered Treasury borrowing cost somewhat".[17] However, the U.S. Treasury experiment yielded inconclusive results: "In a direct comparison of the impact on revenue between the two techniques, the data show a small increase in revenues to the Treasury under the uniform-price technique, but the different

[17] http://www.publicdebt.treas.gov/com/comintro.htm

is not statistically signi¯cant".[18] The Spanish auction mitigates the downward bias on bidding with respect to the discriminatory format, and could increase participation. Our results show that the expected seller's revenue is higher than with discriminatory auctions in a number of cases. If Treasuries are changing from discriminatory to uniform auctions, why not try a hybrid of both that presents good properties?

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Appendix: Notes on Proposition 1.

In this appendix we define some of the notation used in Proposition 1 and present an additional statistical result which helps computation.

Recall that in Proposition 1, x_{-i} is the vector of signals of all bidders but i. Let $g_{\mathbf{x}_{-i}}(x_{-i}/x_i)$ be the density conditional on the signal x_i. We define:

$z^a(x_{-i}; x_i) = g_{\mathbf{x}_{-i}}(x_{-i}/x_i)$;

$z^b(x_{-i}; x_i) = {}_1F_1\left(IK - \left(1^T x_{-i} + x_i\right) + \alpha(1-\mu); IK + \alpha; \rho\right) g_{\mathbf{x}_{-i}}(x_{-i}/x_i)$;

$z^c(x_{-i}; x_i) = z^b(x_{-i}; x_i)$;

$z^d(x_{-i}; x_i) = {}_1F_1\left(IK - \left(1^T x_{-i} + x_i\right) + \alpha(1-\mu); IK + \alpha; 2\rho\right) g_{\mathbf{x}_{-i}}(x_{-i}/x_i)$,

where $1^T x_{-i}$ is the summation of the components of x_{-i} and ${}_1F_1(.)$ is the confluent hypergeometric function (see Abramowitz and Stegun (1972)).

Additionally, it can be proven that:

$g_{\mathbf{x}_{-i}}(x_{-i}/x_i) = M(x_{-i}) \prod_{i=1}^{I-1} \binom{K}{\mathbf{x}_{-i}(j)} \frac{B(x_i + \alpha\mu; K - x_i + \alpha(1-\mu))}{B(\alpha\mu; \alpha(1-\mu))}$

where $M(.)$ and $B(.)$ are the multinomial and beta functions, respectively (see DeGroot (1970)), and $x_{-i}(j)$ is the j-th entry of x_{-i}.

Figures.

Figure 1. An example of equilibrium